Probabilistic Graphical Models

Car Start Problem

In the morning, my car will not start. The start engine turns, but nothing happens. The battery is OK. The problem may be due to dirty spark plugs or the fuel may be stolen. I look at the fuel meter. It shows $\frac{1}{2}$, and I therefore expect the spark plugs to be dirty.

We need to formalize this kind of reasoning:

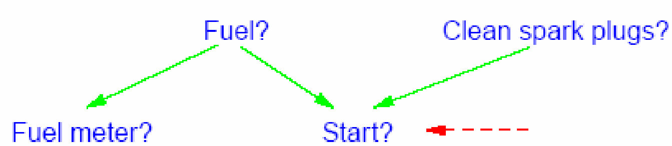
- What made me focus upon fuel and spark plugs?
- Why did I look at the fuel meter?
- Why had fuel meter reading an impact on my belief in dirty spark plugs?

Cause and Effect

Events:

- Fuel?{y,n}
- Clean spark plugs?{y,n}
- Start?{y,n}
- Fuel meter{full, $\frac{1}{2}$, empty}.

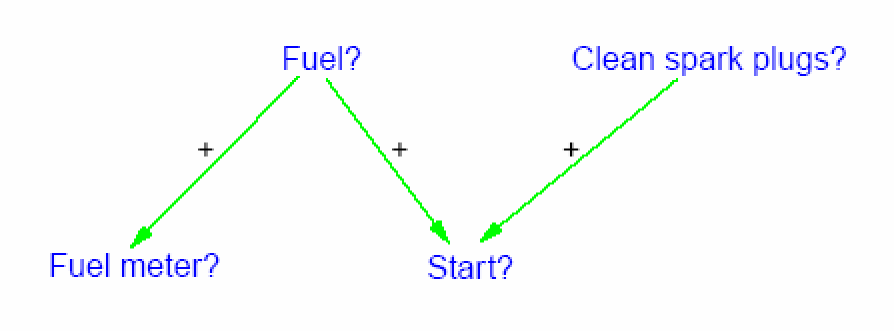
Causal relations:



When I enter the car I have some prior belief on the various events but then start=n.

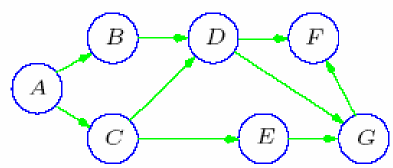
$$\text{Fuel?}_{\to +} \longrightarrow \text{Fuel meter}_{\to +} \qquad \Rightarrow \qquad \text{Fuel?} \to_{+} \text{Fuel meter}$$

$$\boxed{\text{Note: Fuel meter}_{\to +} \Rightarrow \text{Fuel?}_{\to +}}$$



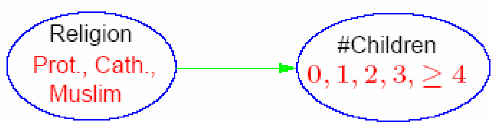
Cause and Effect

A causal network is a directed acyclic graph:



- The nodes are variables with a finite set of states that are mutually exclusive and exhaustive:
  - For example {y,n}, {red, blue, green}, {0,1,2,3,42}.
- The links represent cause – effect relations.

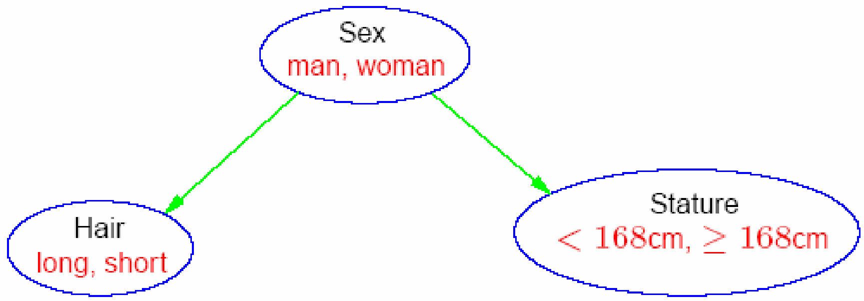
For example:



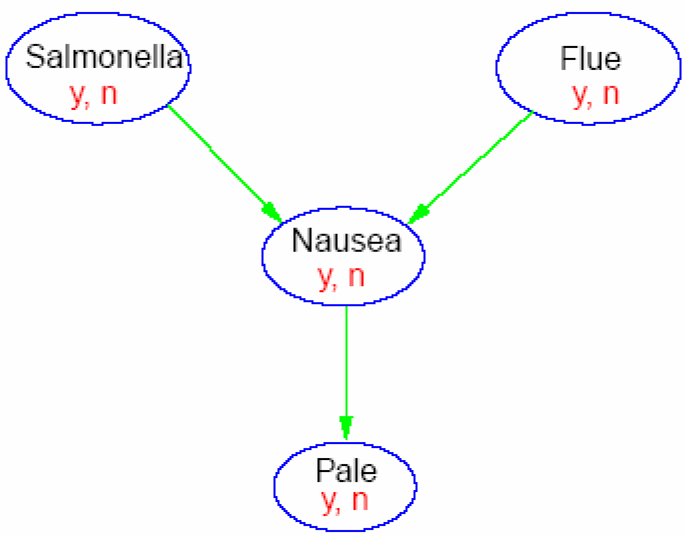
All variables are in exactly one state, but we may not know which one.

Reasoning under Uncertainty

- If there has been a flooding does that tell me something about the amount of rain that has fallen?
- The water level is high: If there has been a flooding does that tell me anything new about the amount of rain that has fallen?

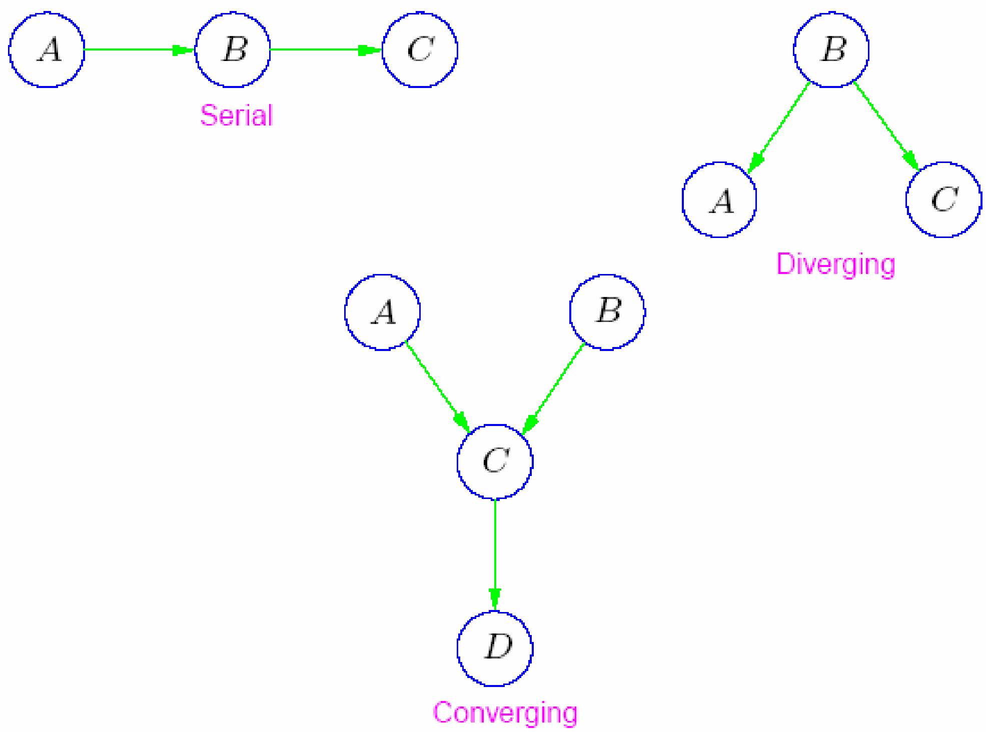


- If a person has long hair does that say something about his/her stature?
- It is a woman: If she has long hair does that say something about her stature?



- Does salmonella have an impact on Flue?
- If a person is Pale, does salmonella then have an impact on Flue?

Relevance changes with evidence



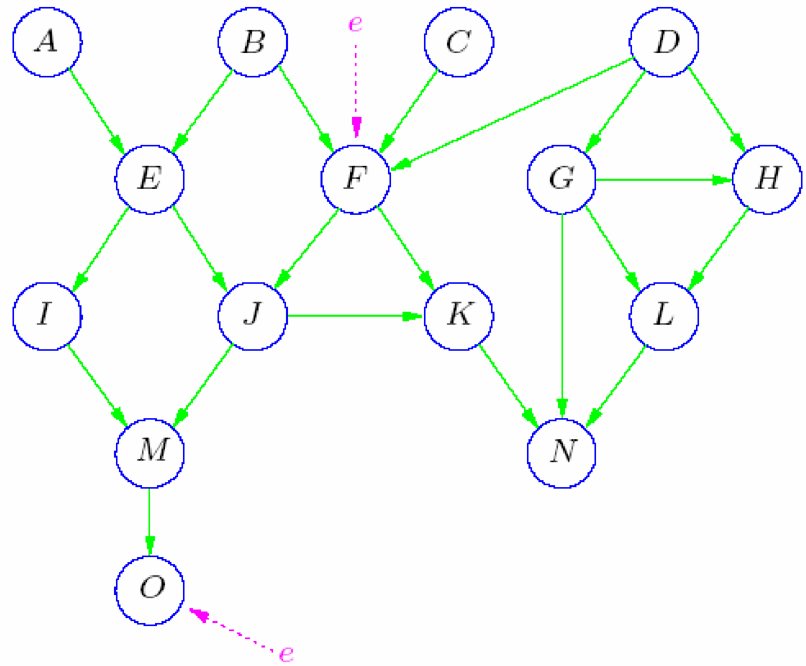
d-Seperation

Definition 2.1 (d-separation). *Two distinct variables A and B in a causal network are* d-separated *("d" for "directed graph") if for all paths between A and B, there is an intermediate variable V (distinct from A and B) such that either*

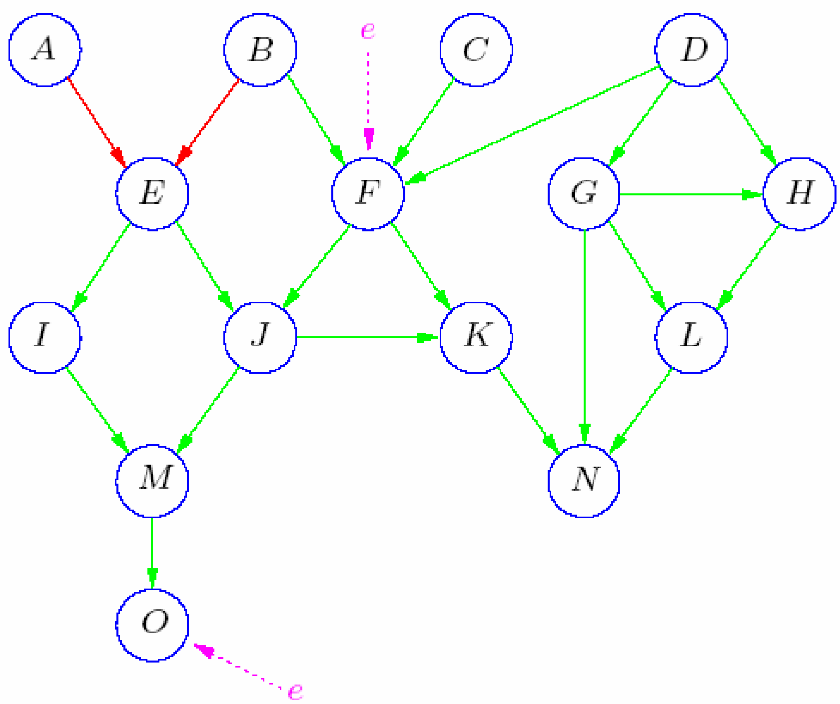
- *the connection is serial or diverging and V is instantiated*
  or
- *the connection is converging, and neither V nor any of V's descendants have received evidence.*

If A and B are not d-separated, we call them d-connected.

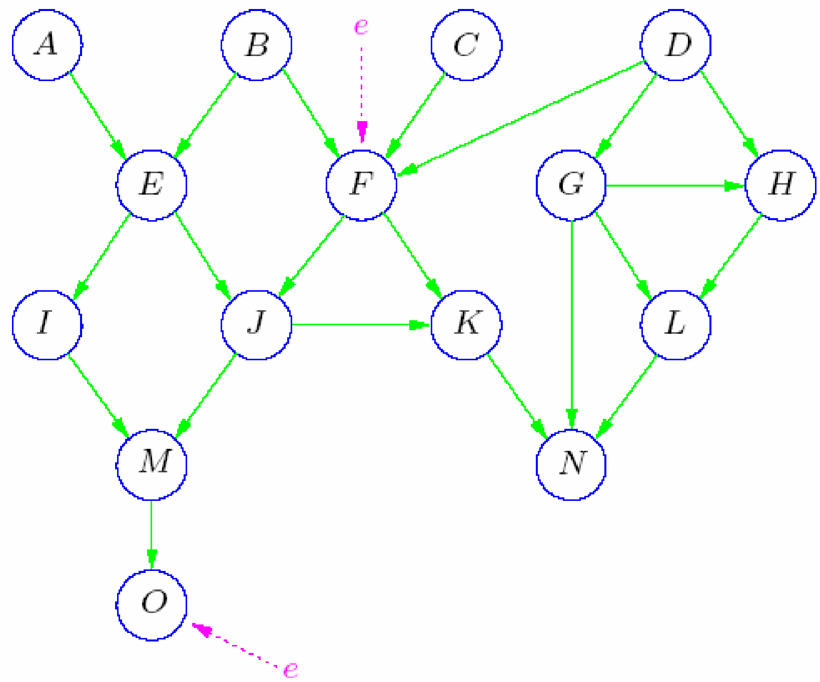
Transmission of Evidence



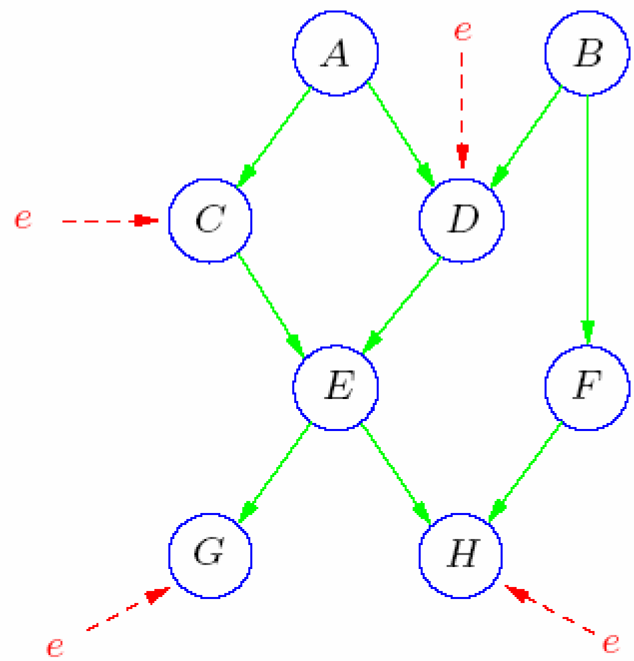
Can knowledge of A have an impact on our knowledge of J?



Can knowledge of A have an impact on our knowledge of B? yes!



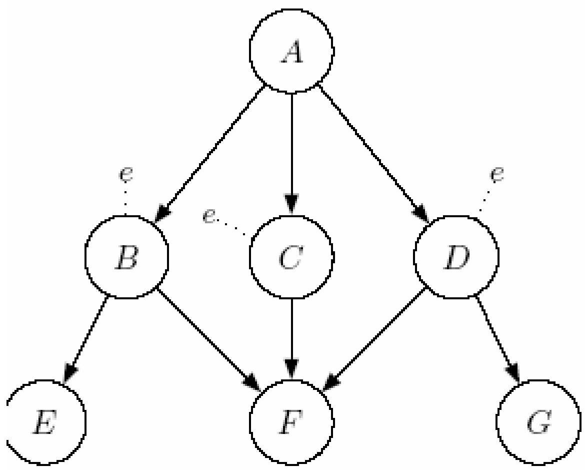
Can knowledge of A have an impact on our knowledge of G?



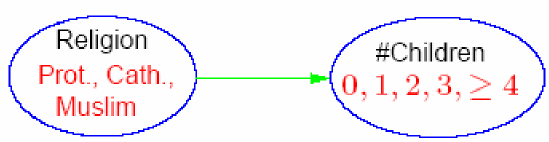
Is E d-separated from A?

Markov Blanket

Definition 2.2. *The* Markov blanket *of a variable A is the set consisting of the parents of A, the children of A, and the variables sharing a child with A.*



Quantification of Causal Networks

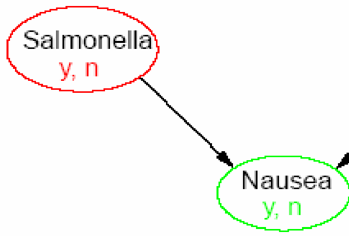


The strength of the link is represented by probabilities:

$$
\begin{array}{lll}
P(0|p) & P(0|c) & P(0|m) \\
P(1|p) & P(1|c) & P(1|m) \\
P(2|p) & P(2|c) & P(2|m) \\
P(3|p) & P(3|c) & P(3|m) \\
P(\geq 4|p) & P(\geq 4|c) & P(\geq 4|m)
\end{array}
\quad \Rightarrow
$$

		Religion		
		p	c	m
#Children	0	0.15	0.05	0.05
	1	0.2	0.1	0.1
	2	0.4	0.2	0.1
	3	0.2	0.4	0.1
	≥ 4	0.05	0.25	0.35

P(#Children|Religion)



$$
\begin{array}{ll}
P(y|y,y) & P(y|y,n) \\
P(n|y,y) & P(n|y,n) \\
P(y|n,y) & P(y|n,n) \\
P(n|n,y) & P(n|n,n)
\end{array}
\quad \Longrightarrow
$$

		Salmonella	
		y	n
Flue	y	$(0.9, 0.1)$	$(0.6, 0.4)$
	n	$(0.8, 0.2)$	$(0.1, 0.9)$

P(Nausea|Salmonella,Flue)

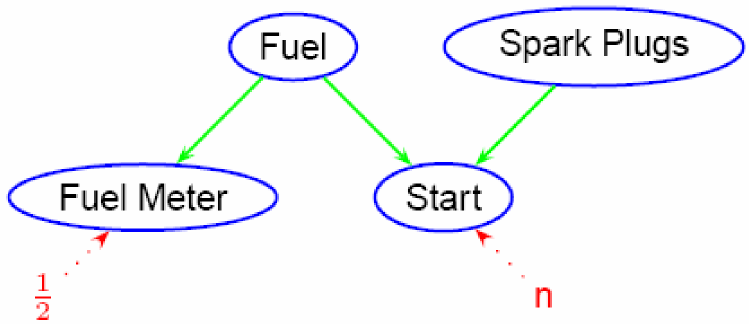
Bayesian Networks

Definition 2.3. *A* Bayesian network *consists of the following:*

- *A set of* variables[1] *and a set of* directed edges *between variables.*
- *Each variable has a finite set of mutually exclusive states.*
- *The variables together with the directed edges form an* acyclic directed graph *(traditionally abbreviated DAG); a directed graph is* acyclic *if there is no directed path $A_1 \to \cdots \to A_n$ so that $A_1 = A_n$.*

- *To each variable A with parents $B_1, \ldots, B_n$, a conditional probability table $P(A \mid B_1, \ldots, B_n)$ is attached.*

Belief Updating in Bayesian Networks



Consider evidence e_1 =(Start=n) and find:

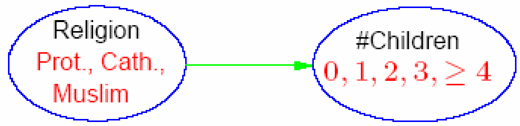
- $P($Spark Plugs$|e_1) = ??$
- $P($Fuel$|e_1) = ??$
- $P($Fuel Meter$|e_1) = ??$

If we also have evidence e_2=(Fuel Meter $= \frac{1}{2}$) what is:

- $P($Spark Plugs$|e_1, e_2) = ??$
- $P($Fuel$|e_1, e_2) = ??$

The basic task for any **probabilistic inference** system is to compute the **posterior probability** distribution for a **set of query** variables, given some observed **event** that is, some assignment of values to a set of evidence variables.

Consider again the network:



Assume that the probabilities are: $P(\text{Religion}) = (0.9_{\text{p}}, 0.04_{\text{c}}, 0.06_{\text{m}})$ and

		Religion		
		p	c	m
#Children	0	0.15	0.05	0.05
	1	0.2	0.1	0.1
	2	0.4	0.2	0.1
	3	0.2	0.4	0.4
	≥ 4	0.05	0.25	0.35

$P(\text{\#Children}|\text{Religion})$

$\Longrightarrow$

		Religion		
		p	c	m
#Children	0	0.135	0.002	0.003
	1	0.18	0.004	0.006
	2	0.36	0.008	0.006
	3	0.18	0.016	0.024
	≥ 4	0.045	0.01	0.021

$P(\text{\#Children}, \text{Religion})$

We want the probability $P(\text{Religion}|\text{\#Children} = 3)$!

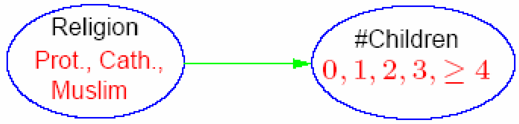
Let A, B and C be variables.

The fundamental rule: $P(A, B) = P(A|B)P(B)$.

Marginalization: $P(A) = \sum_B P(A, B)$

Bayes rule:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$



We can compute $P(\text{Religion}|\#\text{Children} = 3)$ using Bayes' rule:

$$P(\text{Religion}|\#\text{Children} = 3) = \frac{P\left(\#\text{Children} = 3|\text{Religion}\right) P(\text{Religion})}{\sum_{\text{Religion}} P(\text{Religion}, \#\text{Children} = 3)}$$

#Children		Religion p	Religion c	Religion m
	0	0.135	0.002	0.003
	1	0.18	0.004	0.006
	2	0.36	0.008	0.006
	3	0.18	0.016	0.024
	≥ 4	0.045	0.01	0.021

$P(\#\text{Children}, \text{Religion})$

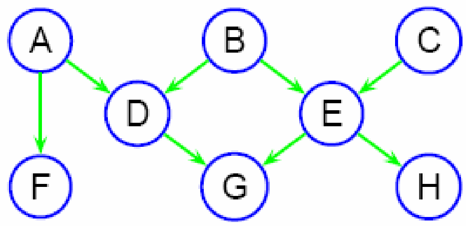
Conditioning

$\Longrightarrow$

	Religion p	Religion c	Religion m
#Ch=3	0.82	0.07	0.11

$P(\text{Religion}|\#\text{Children} = 3)$

Exact Inference Bayesian Networks



Find $P(B|a, f, g, h)$

We can if we have access to $P(a, B, C, D, E, f, g, h)$:

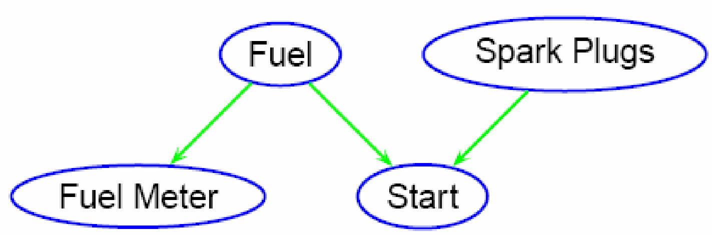
$$P(B, a, f, g, h) = \sum_{C,D,E} P(a, B, C, D, E, f, g, h)$$

$$P(B|a, f, g, h) = \frac{P(B, a, f, g, h)}{P(a, f, g, h)},$$

where

$$P(a, f, g, h) = \sum_{B} P(B, a, f, g, h)$$

Example 1:



	$FM = full$	$FM = \frac{1}{2}$	$FM = empty$
$Sp = yes$	$(0.363, 0)$	$(0.559, 0)$	$(0.0093, 0)$
$Sp = no$	$(0.00015, 0)$	$(0.00024, 0)$	$(3.9 \cdot 10^{-6}, 0)$

Table 2.2. The joint probability table for $P(Fu, FM, SP, St = yes)$.

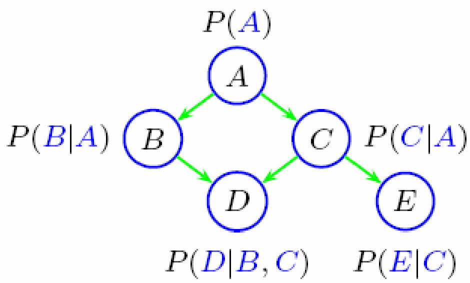
	$FM = full$	$FM = \frac{1}{2}$	$FM = empty$
$Sp = yes$	$(0.00367, 1.9 \cdot 10^{-5})$	$(0.00564, 1.9 \cdot 10^{-5})$	$(9.4 \cdot 10^{-5}, 0.0192)$
$Sp = no$	$(0.01514, 8 \cdot 10^{-7})$	$(0.0233, 8 \cdot 10^{-7})$	$(0.000388, 0.000798)$

Table 2.3. The joint probability table for $P(Fu, FM, SP, St = no)$. The numbers (x, y) in the table represent $(Fu = yes, Fu = no)$.

	$Fu = yes$	$Fu = no$
$Sp = yes$	0.00564	$1.9 \cdot 10^{-5}$
$Sp = no$	0.0233	$8 \cdot 10^{-7}$

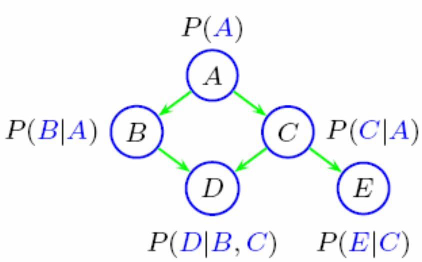
Table 2.4. $P(Fu, SP, St = no, FM = \frac{1}{2})$.

Conditional Probability



Calculate $P(A, B, C, D, E)$

$$
\begin{aligned}
P(A, B, C, D, E) &= P(E|A, B, C, D)P(A, B, C, D) \\
&= P(E|C)P(A, B, C, D) \\
&= P(E|C)P(D|A, B, C)P(A, B, C) \\
&= P(E|C)P(D|B, C)P(A, B, C) \\
&= P(E|C)P(D|B, C)P(C|A, B)P(A, B) \\
&= P(E|C)P(D|B, C)P(C|A)P(B, A) \\
&= P(E|C)P(D|B, C)P(C|A)P(B|A)P(A)
\end{aligned}
$$



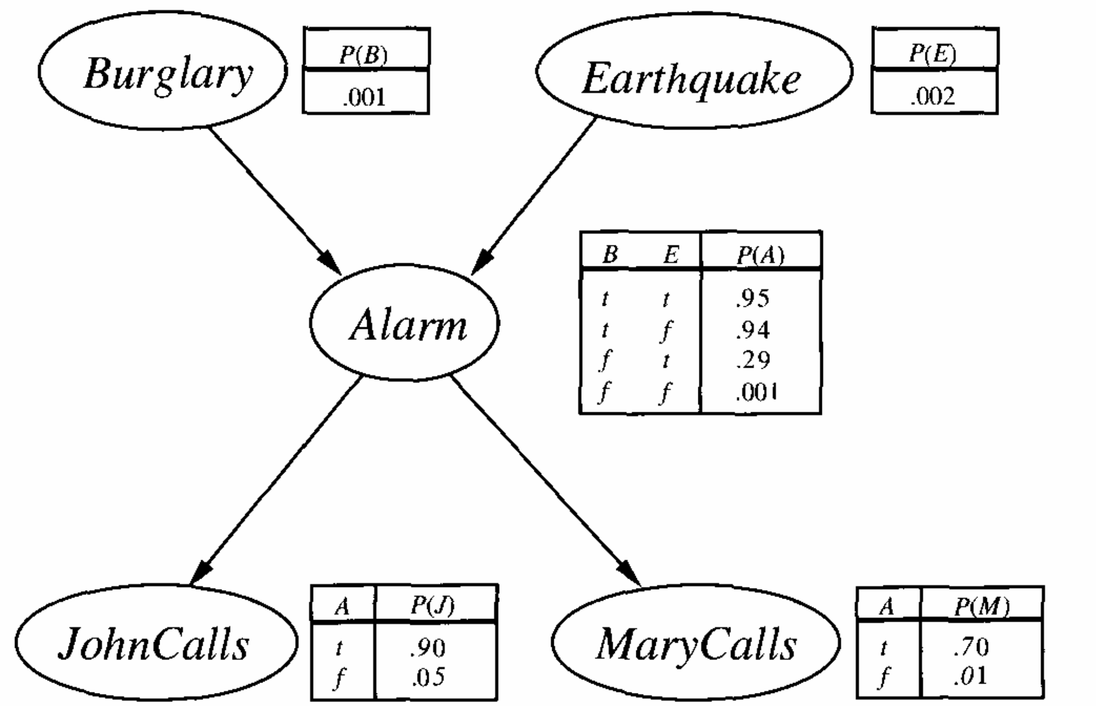
Do we need $P(\mathcal{U}) = P(A, B, C, D, E)$ in order to calculate $P(A|c, e)$?

Note: $P(A|c, e) = \frac{\sum_B \sum_D P(A,B,c,D,e)}{P(c,e)}$.

$$
\begin{aligned}
\sum_B \sum_D P(A, B, c, D, e) &= \sum_B \sum_D P(e|c)P(c|A)P(D|c, B)P(A)P(B|A) \\
&= P(e|c)P(c|A)P(A) \sum_B \sum_D P(D|c, B)P(B|A) \\
&= P(e|c)P(c|A)P(A) \sum_B P(B|A) \sum_D P(D|c, B) \\
&= P(e|c)P(c|A)P(A)
\end{aligned}
$$

So instead of constructing a table with 2^5 entries we only need 2 numbers!

Example 2:



P(B)
.001

P(E)
.002

B	E	P(A)
t	t	.95
t	f	.94
f	t	.29
f	f	.001

A	P(J)
t	.90
f	.05

A	P(M)
t	.70
f	.01